MAVZU. XOS VEKTORLARI BAZIS TASHKIL QILUVCHI CHIZIQLI OPERATORLAR

1. Chiziqli operatorning xos son va xos vektorlari. Xos vektorlari bazis tashkil qiluvchi chiziqli operatorlar. Chiziqli operatorning xos vektorlari bazis tashkil qilishining yetarli sharti.

R^n fazodagi eng sodda chiziqli operatorlar shunday operatorlarki, ular n ta chiziqli erklli vektorga ega.

Haqiqatan, $T : R^n \to R^n$ operator chiziqli erkli $\left\{\vec{e_1}, \vec{e_2}, ..., \vec{e_n}\right\}$ vektorlarga ega bo‘lgan operator bo‘lsin. Shu vektorlarni bazis uchun qabul qilamiz. U holda

$$\left.\begin{array}{l} T(\vec{e_1}) = \lambda_1 \vec{e_1}, \\ T(\vec{e_2}) = \lambda_2 \vec{e_2}, \\ \dots\dots\dots\dots\dots\dots \\ T(\vec{e_n}) = \lambda_n \vec{e_n}, \end{array}\right\}$$

bunda $\lambda_1, \lambda_2, \ldots, \lambda_n$ sonlar T operatorning $\left\{\vec{e_1}, \vec{e_2}, ..., \vec{e_n}\right\}$ xos vektorlariga mos kelgan xos qiymatlari.

Bundan $\left\{\vec{e_1}, \vec{e_2}, ..., \vec{e_n}\right\}$ xos vektorlar tashkil qilgan bazisda T operatorning matritsasi ushbu eng sodda, diagonal ko‘rinishga ega bo‘ladi:

$$A = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ . & . & ... & . \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

Aksincha, agar biror $\left\{\vec{e_1}, \vec{e_2}, ..., \vec{e_n}\right\}$ bazisda T operatorga bunday diagonal matritsa mos kelsa, u holda $\vec{e_1}, \vec{e_2}, ..., \vec{e_n}$ vektorlar T ning xos vektorlari, $\lambda_1, \lambda_2, \ldots, \lambda_n$ esa operatorning $\vec{e_1}, \vec{e_2}, \ldots, \vec{e_n}$ vektorlariga mos keladigan xos qiymatlaridir.

Haqiqatan, A matritsaning xossasidan uning ustunlari $T(\vec{e_1}), T(\vec{e_1}), \ldots, T(\vec{e_n})$ vektorning $\vec{e_1}, \vec{e_2}, \ldots, \vec{e_n}$ bazisdagi komponentlaridan iboratligi kelib chiqadi. Shu sababli

$$T(\vec{e_1}) = \lambda_1 \vec{e_1}, T(\vec{e_2}) = \lambda_2 \vec{e_2}, \ldots, T(\vec{e_n}) = \lambda_n \vec{e_n}.$$

Shuning o‘zi aytilgan tasdiqni isbotlaydi.

1-teorema. Agar R^n da T chiziqli operatorning xos qiymatlari $\lambda_1,\lambda_2,\ldots,\lambda_n$ $(s \le n)$ haqiqiy sonlar to‘plamiga tegishli juft-jufti bilan har xil sonlar bo‘lsa, bu xos qiymatlarga mos keluvchi $\vec{e_1},\vec{e_2},\ldots,\vec{e_s}$ xos vektorlar chiziqli erkli bo‘ladi. Xususan, agar $(s = n)$ bo‘lsa, $\vec{e_1},\vec{e_2},\ldots,\vec{e_n}$ xos vektorlar R^n da bazis tashkil qiladi.

Isbot. Isbotni induksiya metodi bilan olib boriladi. $s=1$ da tasdiqning to‘g‘riligi ravshan. Tasdiq $s-1$ ta vektor uchun o‘rinli deb faraz qilamiz va uni s ta vektor uchun isbotlaymiz. Agar s ta vektor uchun tasdiq to‘g‘rimas deb faraz qilinsa, u holda R da hammasi bir vaqtda nolga teng bo‘lmagan va

$$\gamma_1\vec{e_1} + \gamma_2\vec{e_2} + \ldots\gamma_s\vec{e_s} = 0$$

munosabatni qanoatlantiruvchi

$$\gamma_1,\gamma_2,\ldots,\gamma_s$$

sonlar topiladi. Aniqlik uchun $\gamma_1 \ne 0$ deb faraz qilaylik. Oxirgi tenglikka T operatorni qo‘llanib, quyidagini topamiz:

$$T(\gamma_1\vec{e_1} + \gamma_2\vec{e_2} + \ldots\gamma_3\vec{e_3}) = T(0) = 0,$$

ammo

$$T(\gamma_1\vec{e_1} + \gamma_2\vec{e_2} + \ldots\gamma_s\vec{e_s}) = \gamma_1\lambda_1\vec{e_1} + \gamma_2\lambda_2\vec{e_2} + \ldots\gamma_s\lambda_s\vec{e_s}$$

va shuning uchun

$$\gamma_1\lambda_1\vec{e_1} + \gamma_2\lambda_2\vec{e_2} + \ldots\gamma_s\lambda_s\vec{e_s} = 0$$

Agar oxirgi tenglikdan (*) tenglikni λ_s ga ko‘paytirib ayirilsa, ushbuga ega bo‘lamiz:

$$\gamma_1(\lambda_1 - \lambda_2)\vec{e_1} + \gamma_2(\lambda_2 - \lambda_s)\vec{e_2} + \ldots\gamma_{s-1}(\lambda_{s-1} - \lambda_s)\vec{e}_{s-1} = 0,$$

farazga ko‘ra $\gamma_1 \ne 0$ va $\lambda_1 - \lambda_s \ne 0$ bo‘lgani uchun $\gamma_1(\lambda_1 - \lambda_s) \ne 0$, lekin $(s-1)$ ta $\vec{e_1},\vec{e_2},\ldots,\vec{e}_{s-1}$ vektorlar chiziqli erkli edi. Biz bunda zid natijaga keldik. Demak, induksiya s uchun ham to‘g‘ri ekanini isbot etdik. **Teorema to‘la isbot bo‘ldi.**

1- misol. Shunday $T: R^3 \to R^3$ chiziqli operator berilganki, berilgan tayin $\{\vec{e_1}, \vec{e_2}, \vec{e_3}\}$ bazis uchun T ning matritsasi ushbu ko‘rinishga ega:

$$A = \begin{bmatrix} -1 & 3 & -1 \\ -3 & 5 & -1 \\ -3 & 3 & 1 \end{bmatrix}$$

T operatorning xos sonlari, xos vektorlarini va (agar mumkin bo‘lsa) T operatorning matritsasi diagonal ko‘rinishni oladigan bazisni toping.

Yechish. T operatorning xarakteristik ko‘phadi ushbu ko‘rinishga ega:

$$\det[A-\lambda E]=\begin{vmatrix}-1-\lambda & 3 & -1\\ -3 & 5-\lambda & -1\\ -3 & 3 & 1-\lambda\end{vmatrix}=(\lambda-1)(\lambda-2)^2.$$

Bundan T operatorning xarakteristik sonlari $\lambda=1, \lambda_2=\lambda_3=2$ bo‘ladi. $\lambda_1=1$ songa to‘g‘ri keladigan $q_1=x_1\vec{e_1}+x_2\vec{e_2}+x_3\vec{e_3}$ xos vektor ushbu sistemaning yechimi sifatida topiladi:

$$\left.\begin{aligned}-x_1+3x_2-x_3=x_1,\\ -3x_1+5x_2-x_3=x_2,\\ -3x_1-3x_2+x_3=x_3.\end{aligned}\right\}$$

$q_1=\vec{e_1}+\vec{e_2}+\vec{e_3}$ vektor $\lambda_1=1,$ songa to‘g‘ri keladigan xos vektor ekanini tekshirish oson. $\lambda_2=\lambda_3=2$ xarakteristik sonlarga to‘g‘ri keladigan $\vec{q_2}$ va $\vec{q_3}$ xos vektorlarni topish sistemasi ushbu ko‘rinishga ega:

$$\left.\begin{aligned}-x_1+3x_2-x_3=2x_1,\\ -3x_1+5x_2-x_3=2x_2,\\ -3x_1+3x_2+x_3=2x_3.\end{aligned}\right\}$$

Bevosita tekshirish yo‘li bilan $\vec{q_2}=\vec{e_1}+\vec{e_2}, \vec{q_3}=\vec{e_1}-3\vec{e_3}$ vektorlar T operatorning $\lambda_2=\lambda_3=2$ sonlarga mos xos vektorlari ekaniga ishonch hosil qilamiz. $\vec{q_3}$ vektorlar chiziqli erkli ekanini ko‘rish oson:

$$\begin{vmatrix}1 & 1 & 1\\ 1 & 1 & 0\\ 1 & 0 & -3\end{vmatrix}=-1\neq 0$$

Shu sababli $\vec{q_1},\vec{q_2},\vec{q_3}$ vektorlar bazis tashkil qiladi. $\vec{q_1},\vec{q_2},\vec{q_3}$ lar φ operatorning xos vektorlari bo‘lgani uchun:

$$\left.\begin{aligned}T(\vec{q_1})=\vec{q_1}+0\cdot\vec{q_2}+0\cdot\vec{q_3},\\ T(\vec{q_2})=0\cdot\vec{q_1}+2\cdot\vec{q_2}+0\cdot\vec{q_3},\\ T(\vec{q_3})=0\cdot\vec{q_1}+0\cdot\vec{q_2}+2\vec{q_3}.\end{aligned}\right\}$$

Shu sababli T operatorning $\vec{q_1},\vec{q_2},\vec{q_3}$ bazisdagi matritsasi bunday:

$$\mathrm{B}=\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}$$

2- misol. $T: R^2 \to R^2$ operator $\vec{e_1}, \vec{e_2}$ bazis vektorlarini $T(\vec{e_1}) = \vec{e_1} + i\vec{e_2}$, $\varphi(\vec{e_2}) = \vec{e_2} + i\vec{e_1}$ vektorga o‘tkazuvchi operator bo‘lsin. T operatorning matritsasi diagonal ko‘rinishida bo‘ladigan bazisini topish talab qilinadi.

Yechish. $\vec{e_1}, \vec{e_2}$ bazisda T ning matritsasi ushbu ko‘rinishda bo‘ladi:

$$A = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}.$$

Shunig uchun A operatorning xarakteristik polinomi bunday:

$$\det|(A - \lambda E)| = \begin{vmatrix} 1-\lambda & i \\ i & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - i^2 = \lambda^2 - 2\lambda + 2$$

Ushbu $\lambda_1 = 1 - i$ va $\lambda_2 = 1 + i$ sonlar T operatorning xarakteristik sonlari bo‘ladi. λ_1 va λ_2 xos sonlarga to‘g‘ri keladigan mos $\vec{q_1} = x_1\vec{e_1} + x_2\vec{e_2}$ va $\vec{q_2} = y_1\vec{e_1} + y_2\vec{e_2}$ xos vektorlar quyidagi tenglamalar sistemasidan topiladi:

$$\begin{cases} x_1 + ix_2 = (1-i)x_1 \\ x_1 + ix_2 = (1+i)x_1 \\ ix_1 + x_2 = (1-i)x_2 \\ ix_1 + x_2 = (1+i)x_2 \end{cases}$$

Bundan $\vec{q_1}$ va $\vec{q_2}$ vektorlar sifatida $\vec{q_1} = \vec{e_1} - \vec{e_2}$ va $\vec{q_2} = \vec{e_1} + \vec{e_2}$ chiziqli erkli vektorlarni olish mumkinligi kelib chiqadi, $\vec{q_1}, \vec{q_2}$ bazisda φ vektorning matritsasi ushbu ko‘rinishga ega:

$$\mathrm{B} = \begin{bmatrix} 1-i & 0 \\ 0 & 1+i \end{bmatrix}$$

Chunki

$$\varphi(\vec{q_1}) = (1-i)\vec{q_1},$$
$$\varphi(\vec{q_2}) = (1-i)\vec{q_2}.$$