Mavzu-14. Yuqori tartibli hosila va differensiallar. Differensial hisobning asosiy teoremalari: Ferma, Roll, Lagranj va Koshi teoremalari. Lopital qoidasi. Lagranj formasidagi qoldiq hadli Teylor formulasi. e^x, sinx, cosx, ln(1+x) funksiyalarning Teylor va Makloren formulalari bo'yicha yoyish.

Reja:

1. Yuqori tartibli hosila.
2. Ferma, Roll, Lagranj va Koshi teoremalari.
3. Lopital qoidasi
4. Teylor formulasi.
5. Makloren formulasi.
6. e^x, sinx, cosx, ln(1+x) funksiyalarni Teylor va Makloren formulalari bo'yicha yoyish.

y=f(x) , funksiya R da differensiallanuvchi bo'lsin.

Ta'rif-1 $f'(x)$ funksiyaning hosilasi *f(x)* funksiyaning 2- tartibli hosilasi deyiladi va $f''(x)$ kabi belgilanadi.

Misol $y=\ln\sin\frac{x}{4}$, $y''=?$

$$y'=\left(\ln\sin\frac{x}{4}\right)'=\frac{1}{\sin\frac{x}{4}}\left(\sin\frac{x}{4}\right)'=\frac{\cos\frac{x}{4}}{\sin\frac{x}{4}}\left(\frac{x}{4}\right)'=\frac{1}{4}ctg\frac{x}{4}$$

$$y''=\left(\frac{1}{4}ctg\frac{x}{4}\right)'=\frac{1}{4}\frac{-1}{\sin^2\frac{x}{4}}\left(\frac{x}{4}\right)'=-\frac{1}{16\sin^2\frac{x}{4}}$$

Ta'rif-2 $f^{(n-1)}(x)$ funksiyaning hosilasi *f(x)* funksiyaning n- tartibli hosilasi deyiladi va $f^{(n)}(x)$ kabi belgilanadi.

Misol $y=3^{4x}$, $y^{(5)}=?$

$$y'=\left(3^{4x}\right)'=3^{4x}\ln 3*(4x)'=4\ln 3*3^{4x}$$

$$y''=\left(4\ln 3*3^{4x}\right)'=4\ln 3*\ln 3*3^{4x}*(4x)'=4^2(\ln 3)^2 3^{4x}$$

…

$$y^{(5)}=4^5(\ln 3)^5 3^{4x}$$

Differensial hisobning asosiy teoremalari

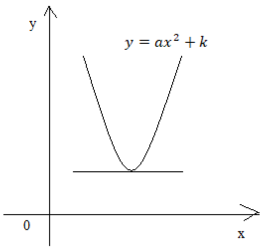
Teorema (Ferma) y=f(x) funksiya berilgan va quyidagi shartlarni qanoatlantirsin:

1. (a;b) intervalda differensiallanuvchi
2. $x_0\in(a;b)$ nuqtada eng katta yoki eng kichik qiymatga erishadi

u holda shu x_0 nuqtada hosila nolga teng, ya'ni $f'(x_0)=0$.

Natija Agar f(x) funksiya (a;b) intervalda eng katta yoki eng kichik qiymatga o'tkazilgan urinma OX o'qiga parallel bo'ladi.

Misol



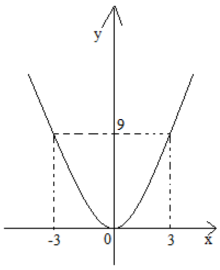
Teorema (Roll) y=f(x) funksiya berilgan va quyidagi shartlarni qanoatlantirsin;

1. [a;b] kesmada uzluksiz;
2. (a;b] oraliqda deifferensiallanuvchi;
3. [a;b] kesmaning chetki nuqalarida qiymatlari teng bo'lsin: f(a)=f(b)

u holda (a;b) oraliqning kamida bitta x_0 nuqtasida $f'(x_0) = 0$

Misol $y = x^2$ funksiya [-3;3] kesmada berilgan bo'lsin.

f(-3)=f(3)=9



Teorema (Lagranj) y=f(x) funksiya berilgan va quyidagi shartlarni qanoatlantirsin:

1. [a;b] kesmadavuzluksiz;
2. (a;b] oraliqda differensiallanuvchi bo'lsin,

u holda (a;b) oraliqda kamida bitta shunday $\exists x_0$, $\frac{f(b)-f(a)}{b-a} = f'(x_0)$

Misol $y = 2x^3 + 3x$, [-2;1] da berilgan

*f(b)=f(1)=2*1+3*1=5*

f(a)=f(-2)=2$(-2)^3$*+3*(-2)=-16-6=-22*

b-a=1-(-2)=3

$$\frac{f(b)-f(a)}{b-a} = \frac{5-(-22)}{3} = 9$$

$$f'(x_0) = \left(2x^3 + 3x\right)' = 6x^2 + 3$$

$6x^2 + 3 = 9$ $\quad x = \pm 1$ $\quad \frac{f(1)-f(-2)}{1-(-2)} = f'(1) = f'(-1)$

Teorema (Koshi) f(x) va g(x) funksiyalar berilgan bo'lsin va quyidagi shartlarni qanoatlantirsin:

1. [a;b] kesmada uzluksiz;

2. (a;b] oraliqda differensiallanuvchi;
3. $g'(x) \neq 0$ (a;b) oraliqda bo'lsin,

u holda $\frac{f(b)-f(a)}{b-a} = \frac{f'(x_0)}{g'(x_0)}$ tenglikni qanoatlantiruvchi kamida bitta x_0 nuqta mavjud.

Misol $f(x) = x^2 + 2x$, $g(x) = 2x^3 + 5x$, [-1;2] da berilgan

$f(b) = f(2) = 2^2 + 2*2 = 8$

$f(a) = f(-1) = 1 + 2*(-1) = -1$

$g(b) = g(2) = 2*2^3 + 5*2 = 16 + 10 = 26$

$g(a) = g(-1) = 2*(-1)^3 + 5*(-1) = -2 - 5 = -7$

$f'(x) = 2x + 2$, $g'(x) = 6x^2 + 5$

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{8-(-1)}{26-(-7)} = \frac{9}{33} = \frac{3}{11}$$

$$\frac{3}{11} = \frac{2x+2}{6x^2+5} \quad , \quad 18x^2 - 22x - 7 = 0$$

$$D = 484 + 4*18*7 = 988 \qquad x_1 = \frac{22-\sqrt{988}}{36} \qquad x_2 = \frac{22+\sqrt{988}}{36}$$

Teorema (Lopital) f(x) va g(x) funksiyalar berilgan bo'lsin va a nuqtaning biror atrofida differensiallanuvchi bo'lsa hamda

1. $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ yoki ∞
2. $g'(x) \neq 0$
3. $\lim_{x\to a} \frac{f(x)}{g(x)}$ mavjud bo'lsa, u holda

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Misol $f(x) = x^2 + 5x$, $g(x) = 3x$

$$\lim_{x\to 0} \frac{x^2+5x}{3x} = \lim_{x\to 0} \frac{2x+5}{3} = \frac{5}{3}$$

y=f(x) funksiya x_0 ning biror $(x_0-\delta, x_0+\delta)$ atrofida aniqlangan va f'(x), f''(x), …, $f^{(n+1)}(x)$ hosilalarga ega va $f^{(n+1)}(x)$ hosila x_0 nuqtada uzluksiz bo'lsin. U holda ushbu :

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''_{(x_0)}}{2!}(x-x_0)^2 + ... + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1} ,$$

Bu yerda $\xi = x_0 + \theta(x - x_0), \quad 0 < \theta < 1$ formula o'rinli bo'ladi.

Agar $\frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1} = R_{n+1}(x)$ belgilash kiritsak :

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''_{(x_0)}}{2!}(x - x_0)^2 + ... + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_{n+1}(x) \qquad (1)$$

hosil bo'ladi. (1) formula **Teylor formulasi**, $R_{n+1}(x)$ esa **Lagranj ko'rinishidagi qoldiq had** deyiladi. n→∞ da $R_{n+1}(x) = 0$ bo'ladi, shuning uchun:

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''_{(x_0)}}{2!}(x - x_0)^2 + ... + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \text{ bo'ladi.}$$

Teylor formulasida $x_0 = 0$ bo'lsa,

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(n)}_{(0)}}{n!}x^n + R_{n+1}(x) \qquad (2)$$

hosil bo'ladi. (2) formula **Makloren formulasi** deyiladi.

Endi e^x, sinx, cosx, ln(1+x) funksiyalar uchun Makloren formulalarini qo'llab yozamiz.

1) $f(x) = e^x$, $f(0) = 1$, $f'(0) = 1$, …. , $f^{(n)}(0) = 1$ ekanligini hisobga olib,

   $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!}$ hosil bo'ladi. n→∞ da $R_{n+1}(x) = 0$

   Xususan, x=1 da $e \approx 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!}$ e sonini taqribiy hisoblash formulasi hosil bo'ladi .

2) $f(x) = \sin x$, $f(0) = 0$, $f^{(n)}(x) = \sin^{(n)}x = \sin(x + n\frac{\pi}{2})$,

   $$f^{(n)}(0) = \sin^{(n)}0 = \sin\frac{\pi n}{2} = \begin{cases} 0, n - juft \\ (-1)^{\frac{n-1}{2}}, n - toq \end{cases}$$

   $$f(x) = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}$$

3) $f(x) = \cos x$, $f(0) = 1$, $f^{(n)}(x) = \cos^{(n)}x = \cos(x + n\frac{\pi}{2})$,

   $$f^{(n)}(0) = \cos\frac{\pi n}{2} = \begin{cases} 0, n - toq \\ (-1)^{\frac{n}{2}}, n - juft \end{cases}$$

   $$f(x) = \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + (-1)^n\frac{x^{2n}}{(2n)!}$$

4) f(x)=ln(1+x) , f(0) = 0 , $f^{(n)}(x)= (-1)^{n+1}\dfrac{(n-1)!}{(1+x)^n}$

Dastlab, n-darajali hosilani topamiz:

$$f'(x) = (\ln(1+x))' = \frac{1}{1+x} = (1+x)^{-1}$$

$$f''(x) = ((1+x)^{-1})' = (-1)(1+x)^{-2}$$

$$f'''(x) = (-1)(-2)(1+x)^{-3}$$

……………………

$$f^{(n)}(x) = (-1)(-2)...(1-n)(1+x)^{-n} = (-1)^{n-1}(n-1)!(1+x)^{-n} = (-1)^{n-1}\frac{(n-1)!}{(1+x)^n}$$

Shunday qilib,

$f^{(n)}(0) = (-1)^{n-1}(n-1)!$. Yuqoridagilarni hisobga olib,

ln(1+x)= $x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \dfrac{x^5}{5} - ... + (-1)^{n-1}\dfrac{x^n}{n}$

Takrorlash uchun savollar:

1. $f(x) = 3x^4 + 2x^3 + 3x$ $f''(x) = ?$

2. $f(x) = \sin x \cos^2 x$ $f''(x) = ?$

3. $y = e^x$ $y^{(6)} = ?$

4. $\lim\limits_{x\to 0}\dfrac{5x^2+2x}{4x} = ?$

5. $\lim\limits_{x\to 0}\dfrac{6x^2-5x}{3x+1} = ?$

6. Teylor formulasini ifodalang

7. Makloren formulasini ifodalang

8. e^x funksiya uchun Makloren formulasini yozing.
9. sinx funksiya uchun Makloren formulasini yozing.
10. cosx funksiya uchun Makloren formulasini yozing.
11. ln(1+x) funksiya uchun Makloren formulasini yozing.

Foydalanilgan adabiyotlar:

1. Oliy matematika. (F.Rajabov) 2007-y. Toshkent.
2. Курс математического анализа. (Л. Кудрявцев) 1998-г. Москва.
3. www.mathprofi.ru internet sayti.